The Mandelbrot Set is Universal

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Abstract

We show small Mandelbrot sets are dense in the bifurcation locus for any holomorphic family of rational maps.

1 Introduction

Fix an integer $d \geq 2$, and let $p_c(z) = z^d + c$. The *generalized Mandelbrot set* $M_d \subset \mathbb{C}$ is defined as the set of c such that the Julia set $J(p_c)$ is connected. Equivalently, $c \in M_d$ iff $p_c^n(0)$ does not tend to infinity as $n \to \infty$. The traditional Mandelbrot set is the quadratic version M_2.

A *holomorphic family of rational maps over* X is a holomorphic map

$$f : X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

where X is a complex manifold and $\widehat{\mathbb{C}}$ is the Riemann sphere. For each $t \in X$ the family f specializes to a rational map $f_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, denoted $f_t(z)$. For convenience we will require that X is *connected* and that $\deg(f_t) \geq 2$ for all t.

The *bifurcation locus* $B(f) \subset X$ is defined equivalently as the set of t such that:

1. The number of attracting cycles of f_t is not locally constant;
2. The period of the attracting cycles of f_t is locally unbounded; or
3. The Julia set $J(f_t)$ does not move continuously (in the Hausdorff topology) over any neighborhood of t.

It is known that $B(f)$ is a closed, nowhere dense subset of X; its complement $X - B(f)$ is also called the *J-stable set* [MSS], [Mc2, §4.1].

As a prime example, $p_c(z) = z^d + c$ is a holomorphic family parameterized by $c \in \mathbb{C}$, and its bifurcation locus is ∂M_d. See Figure 1.

In this paper we show that *every* bifurcation set contains a copy of the boundary of the Mandelbrot set or its degree d generalization. The Mandelbrot sets M_d are thus *universal*; they are initial objects in the category

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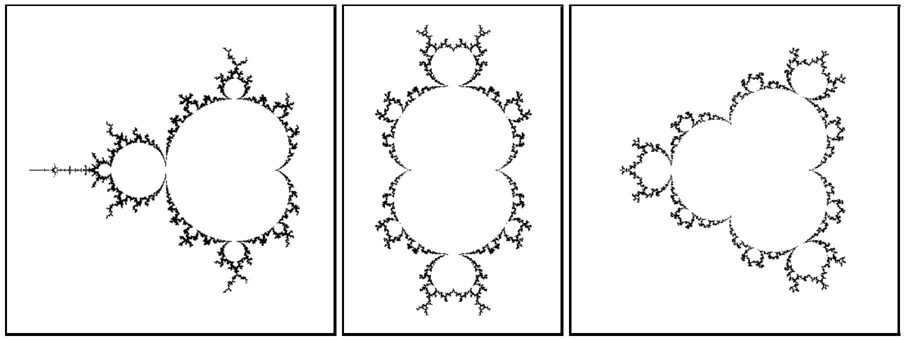

Figure 1. Mandelbrot sets of degrees 2, 3 and 4.

of bifurcations, providing a lower bound on the complexity of $B(f)$ for all families f_t.

For simplicity we first treat the case $X = \Delta = \{t : |t| < 1\}$.

Theorem 1.1 *For any holomorphic family of rational maps over the unit disk, the bifurcation locus $B(f) \subset \Delta$ is either empty or contains the quasiconformal image of ∂M_d for some d.*

The proof (§4) shows that $B(f)$ contains copies of ∂M_d with arbitrarily small quasiconformal distortion, and controls the degrees d that arise. For example we can always find a copy of ∂M_d with $d \leq 2^{2\deg(f_t)-2}$, and generically $B(f)$ contains a copy of ∂M_2 (see Corollary 4.4). Since the Theorem is local we have:

Corollary 1.2 *Small Mandelbrot sets are dense in $B(f)$.*

There is also a statement in the dynamical plane:

Theorem 1.3 *Let f be a holomorphic family of rational maps with bifurcations. Then there is a $d \geq 2$ such that for any $c \in M_d$ and $m > 0$, the family contains a polynomial-like map $f_t^n : U \to V$ hybrid conjugate to $z^d + c$ with* $\mathrm{mod}(U - V) > m$.

Corollary 1.4 *If f has bifurcations then for any $\epsilon > 0$ there exists a t such that $f_t(z)$ has a superattracting basin which is a $(1+\epsilon)$-quasidisk.*

Proof. The family contains a polynomial-like map $f_t^n : U \to V$ hybrid conjugate to $p_0(z) = z^d$, a map whose superattracting basin is a round disk. Since $\mathrm{mod}(V - U)$ can be made arbitrarily large, the conjugacy can be made nearly conformal, and thus f_t has a superattracting basin which is a $(1+\epsilon)$-quasidisk. ∎

For applications to Hausdorff dimension we recall:

Theorem 1.5 (Shishikura) *For any $d \geq 2$, the Hausdorff dimension of ∂M_d is two. Moreover* $\mathrm{H.dim}(J(p_c)) = 2$ *for a dense G_δ of $c \in \partial M_d$.*

This result is stated for $d = 2$ in [Shi2] and [Shi1] but the argument generalizes to $d \geq 2$. Quasiconformal maps preserve sets of full dimension [GV], so from Theorems 1.1 and 1.3 we obtain:

Corollary 1.6 *For any family of rational maps f over Δ, the bifurcation set $B(f)$ is empty or has Hausdorff dimension two.*

Corollary 1.7 *If f has bifurcations, then* $\mathrm{H.dim}(J(f_t)) = 2$ *for a dense set of $t \in B(f)$.*[1]

For higher-dimensional families one has (§5):

Corollary 1.8 *For any holomorphic family of rational maps over a complex manifold X, either $B(f) = \emptyset$ or* $\mathrm{H.dim}(B(f)) = \mathrm{H.dim}(X) = 2\dim_{\mathbb{C}} X$.

Similar results on Hausdorff dimension were obtained by Tan Lei, under a technical hypothesis on the family f [Tan].

A family of rational maps f is *algebraic* if its parameter space X is a quasi-projective variety (such as $\mathbb{C}^n$) and the coefficients of $f_t(z)$ are rational functions of t. For example, $p_c(z) = z^d + c$ is an algebraic family over $X = \mathbb{C}$. Such families almost always contain bifurcations [Mc1]:

Theorem 1.9 *For any algebraic family of rational maps, either*

1. *The family is trivial (f_t and f_s are conformally conjugate for all $t, s \in X$); or*

2. *The family is affine (every f_t is critically finite and double covered by a torus endomorphism); or*

3. *The family has bifurcations ($B(f) \neq \emptyset$).*

Corollary 1.10 *With rare exceptions, any algebraic family of rational maps exhibits small Mandelbrot sets in its parameter space.*

[1] This set of t can be improved to a dense G_δ using Shishikura's idea of hyperbolic dimension.

This Corollary was our original motivation for proving Theorem 1.1.

As another application, for $t \in \mathbb{C}^{d-1}$ let

$$f_t(z) = z^d + t_1 z^{d-2} + \cdots + t_{d-1}$$

and let

$$\mathcal{C}_d = \{t \,:\, J(f_t) \text{ is connected}\}$$

denote the *connectedness locus*. Then we have:

Corollary 1.11 (Tan Lei) *The boundary of the connectedness locus has full dimension; that is,* $\mathrm{H.}\dim(\partial\mathcal{C}_d) = \mathrm{H.}\dim(\mathcal{C}_d) = 2d-2$.

Proof. Consider the algebraic family $g_a(z) = z^d + az^{d-1}$, which for $a \neq 0$ has all but one critical point fixed under g_a. By Theorem 1.9, this family has bifurcations at some $a \in \mathbb{C}$. Then there is a neighborhood U of $(a, 0, \ldots, 0) \in \mathbb{C}^{d-2}$ such that for $t \in U$ all critical points of f_t save one lie in an attracting or superattracting basin. If $t \in B(f) \cap U$, then the remaining critical point has a bounded forward orbit under f_t, but under a small perturbation tends to infinity. It follows that $B(f) \cap U = \partial\mathcal{C}_d \cap U \neq \emptyset$, and thus $\dim(\partial\mathcal{C}_d) \geq \dim B(f) = 2d-2$. ■

Remark. Rees has shown that the bifurcation locus has positive measure in the space of all rational maps of degree d [Rs]; it would be interesting to known general conditions on a family f such that $B(f)$ has positive measure in the parameter space X.

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2 Families of rational maps

In this section we begin a more formal study of maps with bifurcations.

Definitions. A *local bifurcation* is a holomorphic family of rational maps $f_t(z)$ over the unit disk Δ, such that $0 \in B(f)$.

The following natural operations can be performed on f to construct new local bifurcations:

1. *Coordinate change:* replace f_t by $m_t \circ f_t \circ m_t^{-1}$, where $m : \Delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic family of Möbius transformations.

2. *Iteration:* replace $f_t(z)$ by $f_t^n(z)$ for a fixed $n \geq 1$.

3. *Base change:* replace $f_t(z)$ by $f_{\phi(t)}(z)$, where $\phi : \Delta \to \Delta$ is a nonconstant holomorphic map with $\phi(0) \in B(f)$.

The first two operations leave the bifurcation locus unchanged, while the last transforms $B(f)$ to $\phi^{-1}(B(f))$.

Marked critical points. We will also consider pairs (f, c) consisting of a local bifurcation and a *marked critical point*; this means $c : \Delta \to \widehat{\mathbb{C}}$ is holomorphic and $f_t'(c_t) = 0$. The operations above also apply to (f, c); a coordinate change replaces c_t with $m_t(c_t)$ and a base change replaces c_t with $c_{\phi(t)}$.

Misiurewicz points. A marked critical point c of f is *active* if its forward orbit

$$\langle f_t^n(c_t) \; : \; n = 1, 2, 3, \ldots \rangle$$

fails to form a normal family of functions of t on any neighborhood of $t = 0$ in Δ. A parameter t is a *Misiurewicz point* for (f, c) if the forward orbit of c_t under f_t lands on a repelling periodic cycle. If $t = 0$ is a Misiurewicz point, then either c is active or c_t is preperiodic for all t.

Proposition 2.1 *If c is an active critical point, then (f, c) has a sequence of distinct Misiurewicz points $t_n \to 0$.*

Proof. This is a traditional normal families argument. Choose any 3 distinct repelling periodic points $\{a_0, b_0, c_0\}$ for f_0, and let $\{a_t, b_t, c_t\}$ be holomorphic functions parameterizing the corresponding periodic points of f_t for t near zero. Since $\langle f_t^n(c_t)\rangle$ is not a normal family, it cannot avoid these three points, and any parameter t where $f_t^n(c_t)$ meets a_t, b_t or c_t is a Misiurewicz point. ∎

Ramification. Next we discuss the existence of univalent inverse branches for a single rational map $F(z)$. Let $d = \deg(F, z)$ denote the local degree of F at $z \in \widehat{\mathbb{C}}$; we have $d > 1$ iff z is a critical point of multiplicity $(d-1)$. We say y is an *unramified preimage* of z if for some $n \geq 0$, $F^n(y) = z$ and $\deg(F^n, y) = 1$. We say z is *unramified* if it has infinitely many unramified preimages. In this case its unramified preimages accumulate on the full Julia set $J(F)$.

Proposition 2.2 *If z has 5 distinct unramified preimages then it has infinitely many.*

Proof. Let E be the set of all unramified preimages of z, and let C be the critical points of F. Then $F^{-1}(E) \subset E \cup C$, so if $|E|$ is finite then

$$d|E| = \sum_{z \in F^{-1}(E)} 1 + \operatorname{mult}(f', z) \leq |F^{-1}(E)| + 2d - 2 \leq |E| + 4d - 4$$

and therefore $|E| \leq 4$. ∎

Corollary 2.3 *Let (f, c) be a local bifurcation with marked critical point. Then the set of t such that c_t is ramified for f_t is either discrete or the whole disk.*

Proof. By the previous Proposition, the ramified parameters are defined by a finite number of analytic equations in t. ■

Proposition 2.4 *After a suitable base change, any local bifurcation f can be provided with an active marked critical point c such that c_0 is unramified for f_0.*

Remark. It is possible that all the active critical points are ramified at $t = 0$. The base change in the Proposition will generally not preserve the central fiber f_0.

Proof. The set $C = \{(t, z) \in \Delta \times \widehat{\mathbb{C}} \,:\, f_t'(z) = 0\}$ is an analytic variety with a proper finite projection to Δ. By Puiseux series, after a base change of the form $\phi(t) = \epsilon t^n$ all the critical points of f can be marked by holomorphic functions $\{c_t^1, \ldots, c_t^m\}$. Since $t = 0$ is in the bifurcation set, by [Mc2, Thm. 4.2], there is an i such that $\langle f_t^n(c_t^i)\rangle$ is not a normal family at $t = 0$. That is, c^i is an active critical point.

Next we show c^i can be chosen so that for generic t it is disjoint from the forward orbits of all other critical points. If not, there is a c^j and $n \geq 1$ such that $f_t^n(c_t^j) = c_t^i$ for all t. Then c^j is also active and we may replace c^i with c^j. If the replacement process were to cycle, then c^i would be a periodic critical point, which is impossible because it is active. Thus we eventually achieve a c^i which is generically disjoint from the forward orbits of the other critical points.

In particular, there is a t such that c_t^i is unramified for f_t. By Corollary 2.3, the set $R \subset \Delta$ of parameters where c_t^i is ramified is discrete. By Proposition 2.1, there are Misiurewicz points t_n for (f, c^i) with $t_n \to 0$. Choose n such that $t_n \notin R$, and make a base change moving t_n to zero; then c^i is active, and c_0^i is unramified for f_0. ■

Misiurewicz bifurcations. Let (f, c) be a local bifurcation with a marked critical point. We say (f, c) is a *Misiurewicz bifurcation* of degree d if

M1. $f_0(c_0)$ is a repelling fixed-point of f_0;

M2. c_0 is unramified for f_0;

M3. $f_t(c_t)$ is not a fixed-point of f_t, for some t; and

M4. $\deg(f_t, c_t) = d$ for all t sufficiently small.

Proposition 2.5 *For any local bifurcation (f, c) with c active and c_0 unramified, there is a base change and an $n > 0$ such that (f^n, c) is a Misiurewicz bifurcation.*

Remark. The delicate point is condition (M4). The danger is that for every Misiurewicz parameter t, the forward orbit of c_t might accidentally collide with another critical point before reaching the periodic cycle. We must avoid these collisions to make the degree of f_t^n at c_t locally constant.

Proof. There are Misiurewicz points $t_n \to 0$ for (f, c), and c_t is unramified for all t near 0, so after a base change and replacing f with f^n we can arrange that (f, c) satisfies conditions (M1), (M2) and (M3).

We can also arrange that $\deg(f_t, c_t) = d$ for all $t \neq 0$. However (M4) may fail because $\deg(f_t, c_t)$ may jump up at $t = 0$. This jump would occur if another critical of f_t coincides with c_t at $t = 0$.

To rule this out, we make a further perturbation of f_0. Let a_t locally parameterize the repelling fixed-point of f_t such that $f_0(c_0) = a_0$. Choose a neighborhood U of a_0 such that for t small, f_t is linearizable on U and U is disjoint from the critical points of f_t. (This is possible since $f_0'(a_0) \neq 0$.)

Let $b_t \in U - \{a_t\}$ be a parameterized repelling periodic point close to a_t. Then b_t has preimages in U accumulating on a_t. Choose s near 0 such that $f_s(c_s)$ hits one of these preimages (such an s exists by the argument principle and (M3)). For this special parameter, c_s first maps close to a_s, then remains in U until it finally lands on b_s. Since there are no critical points in U, we have $\deg(f_s^i, c) = d$ for all $i > 0$.

Making a base change moving s to $t = 0$, we find that (f^n, c) satisfies (M1-M4) for n a suitable multiple of the period of b_s. ∎

3 The Misiurewicz cascade

In this section we show that when a Misiurewicz point bifurcates, it produces a cascade of polynomial-like maps.

Definitions. A *polynomial-like map* $g : U \to V$ is a proper, holomorphic map between simply-connected regions with $\overline{U}$ compact and $\overline{U} \subset V \subset \mathbb{C}$ [DH]. Its *filled Julia set* is defined by

$$K(g) = \bigcap_1^\infty g^{-n}(V);$$

it is the set of points that never escape from U under forward iteration.

Any polynomial such as $p_c(z) = z^d + c$ can be restricted to a polynomial-like map $p_c : U \to V$ of degree d with the same filled Julia set. Moreover small analytic perturbations of $p_c : U \to V$ are also polynomial-like.

A degree d Misiurewicz bifurcation (f, c) gives rise to polynomial-like maps $f_t^n : B_0 \to B_n$, by the following mechanism. For small t, a small ball B_0 about the critical point c_t maps to a small ball B_1 close to, but not containing, the fixed-point of f_t. The iterates $B_i = f_t^i(B)$ then remain near the fixed-point for a long time, ultimately expanding by a large factor. Finally for suitable t, as B_i escapes from the fixed-point it maps back over B_0, resulting in a degree d map $f_t^n : B_0 \to B_n \supset B_0$. Since most of the images $\langle B_i \rangle$ lie in the region where f_t behaves linearly, the first-return map $f_t^n : B_0 \to B_n$ behaves like a polynomial of degree d.

This scenario leads to a cascade of families of polynomial-like maps, indexed by the return time n. Here is a precise statement.

Theorem 3.1 *Let (f, c) be a degree d Misiurewicz bifurcation, and fix $R > 0$. Then for all $n \gg 0$, there is a coordinate change depending on n such that $c_t = 0$ and*

$$f_t^n(z) = z^d + \xi + O(\epsilon_n)$$

whenever $|z|, |\xi| \leq R$. Here $t = t_n(1 + \gamma_n \xi)$, t_n and γ_n are nonzero, and γ_n, t_n and ϵ_n tend to zero as $n \to \infty$.

The constants in $O(\cdot)$ above depend on f and R but not on n.

The proof yields more explicit information. Let $\lambda_0 = f_0'(f_0(c_0))$ be the multiplier of the fixed-point on which c_0 lands, and let r be the multiplicity of intersection of the graph of c_t and the graph of this fixed-point at $t = 0$. Then for $t = t_n$, the critical point c_t is periodic with period n, and we have:

$$\begin{aligned} t_n &\sim C\lambda_0^{-n/r}, & (3.1)\\ \gamma_n &= C'\lambda_0^{-n/(d-1)}, \quad \text{and} & (3.2)\\ \epsilon_n &= n(|\lambda_0|^{-n/r} + |\lambda_0|^{-n/(d-1)}), & (3.3) \end{aligned}$$

for certain constants C, C' depending on f. Due to the choice of roots, there are r possibilities for t_n and $(d-1)$ for γ_n; the Theorem is valid for all choices. Finally for ξ fixed and $t = t_n(1 + \gamma_n \xi)$, the map f_t^n is polynomial-like near c_t for all $n \gg 0$, and in the *original* z-coordinate its filled Julia set satisfies

$$\operatorname{diam} K(f_t^n) \asymp |\lambda_0|^{-n/(d-1)}.$$

Notation. We adopt the usual conventions: $a_n = O(b_n)$, $a_n \asymp b_n$, $a_n \sim b_n$ and $n \gg 0$ mean $|a_n| < C|b_n|$, $(1/C)|b_n| < |a_n| < C|b_n|$, $a_n/b_n \to 1$ and $n \geq N$, where C and N are implicit constants.

Proof. We will make several constructions that work on a small neighborhood of $t = 0$. First, let a_t parameterize the repelling fixed-point of f_t such that $a_0 = f_0(c_0)$. Let $\lambda_t = f_t'(a_t)$ be its multiplier. There is a holomorphically varying coordinate chart $u = \phi_t(z)$ defined near $z = a_t$ such that

$$\phi_t \circ f_t \circ \phi_t^{-1}(u) = \lambda_t u \tag{3.4}$$

for u near 0. We call $u = \phi_t(z)$ the *linearizing coordinate*; note that $u = 0$ at a_t.

We next arrange that $u = 1$ is an unramified preimage of c_t. Since c_0 is unramified by (M2), its unramified preimages accumulate on a_0. Let b_0 be one such preimage, with $f_0^p(b_0) = c_0$ and b_0 in the domain of ϕ_0. Then b_0 prolongs to a holomorphic function b_t with $f_t^p(b_t) = c_t$. Replacing ϕ_t by $\phi_t(z)/\phi_t(b_t)$, we can assume $u = \phi_t(b_t) = 1$.

For small t, the composition $f_t^p \circ \phi_t^{-1}$ is univalent near $u = 1$. By applying a coordinate change $z \mapsto m_t(z)$, where m_t is a Möbius transformation depending on t, we can arrange that $c_t = 0$ and that

$$f_t^p \circ \phi_t^{-1}(u) = (u-1) + O((u-1)^2) \tag{3.5}$$

on $B(1, \epsilon)$.

Since $\deg(f_t, 0) = d$ for t near 0 by (M4), we have

$$\begin{aligned} \phi_t \circ f_t(z) &= \sum A_i(t) z^i & (3.6)\\ &= A_0(t) + A_d(0) z^d \left(1 + O(|z| + |t|)\right) & (3.7)\end{aligned}$$

with $A_d(0) \neq 0$. Here $A_0(t) = f_t(0)$ is the u-coordinate of the critical value. By (M3), c_t is not pre-fixed for all t, so there is an $r > 0$ such that

$$A_0(t) = t^r B(t) \tag{3.8}$$

where $B(0) \neq 0$.

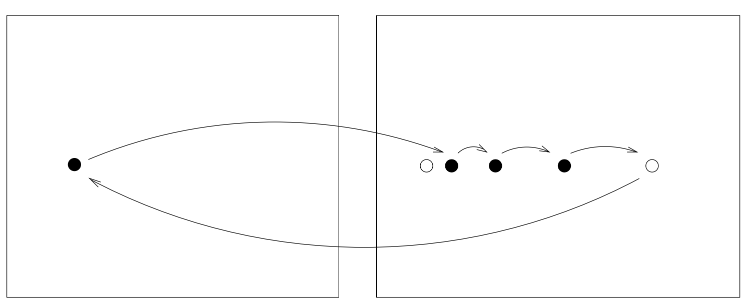

Figure 2. Visiting the repelling fixed-point

Next for $n \gg 0$ we choose t_n such that

$$f_t^{1+n+p}(c_t) = c_t \quad \text{when } t = t_n. \tag{3.9}$$

More precisely, for $t = t_n$ we will arrange that c_t maps first close to a_t, then lands after n iterates on b_t, and thus returns in p further iterates to c_t; see Figure 2. In the u-coordinate system, f_t is linear and $b_t = 1$, so the equation $f_t^{n+1}(c_t) = b_t$ becomes

$$\lambda_t^n A_0(t) = 1 \quad \text{when } t = t_n. \tag{3.10}$$

By the argument principle, for $n \gg 0$ this equation has a solution t_n close to any root of the approximation $\lambda_0^n t^r B(0) = 1$ obtained from (3.8). Moreover

$$t_n \sim B(0)^{-1}\lambda_0^{-n/r}$$

(verifying (3.1)), and t_n satisfies (3.9) because $f_t^p(b_t) = c_t$. (There are actually be r solutions for t_n for a given n; any one of the r solutions will do.)

We now turn to the estimate of $f_t^{1+n+p}(z)$ for (t, z) near $(t_n, 0)$. We will assume throughout that $t = t_n + s$ and that:

$$|z| \textit{ and } |s/t_n| \textit{ are } O(\Lambda^{-n/(d-1)}) \tag{3.11}$$

where $\Lambda = |\lambda_0| > 1$. (To see this is the correct scale at which to work, suppose $\operatorname{diam}(B) \asymp \operatorname{diam} f_t^{1+n+p}(B)$, where B is a ball centered at $z = 0$. Then $\operatorname{diam} f_t(B) \asymp (\operatorname{diam} B)^d$, and f_t^n is expanding by a factor of about Λ^n, while f_t^p is univalent, so we get $\operatorname{diam} B \asymp \Lambda^n(\operatorname{diam} B)^d$, or $\operatorname{diam} B \asymp \Lambda^{-n/(d-1)}$. Similarly $|f_t^{1+n+p}(0)| \asymp \Lambda^n(s/t_n)t_n^r \asymp (s/t_n) = O(\operatorname{diam} B)$ when s is as above.)

It is also convenient to set

$$\widetilde{\Lambda} = \min(\Lambda^{1/(d-1)}, \Lambda^{1/r}) > 1,$$

so that we may assert:

$$z \textit{ and } t \textit{ are } O(\widetilde{\Lambda}^{-n}). \tag{3.12}$$

By (3.11) the n iterates of $f_t(z)$ lie within the domain of linearization, so by (3.7) we have

$$\phi_t \circ f_t^{1+n}(z) = \lambda_t^n A_0(t) + \lambda_t^n A_0(d) z^d (1 + O(|z| + |t|)).$$

The first term is approximately 1. Indeed, $\lambda_t^n = \lambda_{t_n}^n(1 + O(ns))$, so by (3.8) we have

$$\begin{aligned}
\lambda_t^n A_0(t) &= \lambda_t^n (t_n + s)^r B(t_n + s) \\
&= \lambda_{t_n}^n (1 + O(ns)) \cdot t_n^r \left(1 + \frac{s}{t_n}\right)^r \cdot B(t_n)(1 + O(s)) \\
&= \lambda_{t_n}^n A_0(t_n) \left(1 + r\frac{s}{t_n} + O((s/t_n)^2) + O(ns)\right) \\
&= 1 + r\frac{s}{t_n} + O((s/t_n)^2) + O(ns)
\end{aligned}$$

by (3.10). Similarly, $\lambda_t^n = \lambda_0^n(1 + O(t))$, so

$$\begin{aligned}
&\phi_t \circ f_t^{1+n}(z) - 1 = \\
&\lambda_0^n A_0(d) z^d (1 + O(|z| + |nt|)) + r\frac{s}{t_n} + O((s/t_n)^2) + O(ns) = \\
&\lambda_0^n A_0(d) z^d + r\frac{s}{t_n} + O(n\Lambda^{-n/(d-1)}\widetilde{\Lambda}^{-n}),
\end{aligned}$$

using (3.11) and (3.12). The expression above as a whole is $O(\Lambda^{-n/(d-1)})$, so composing with the univalent map $f_t^p \circ \phi_t^{-1}$ introduces (by (3.5)) an additional error of size $O(\Lambda^{-2n/(d-1)})$, which is already accounted for in the $O(\cdot)$ above. Thus the expression above also represents $f_t^{1+n+p}(z)$.

Finally we make a linear change of coordinates of the form $z \mapsto \alpha_n z$, conjugating the expression above to

$$f_t^{1+n+p}(z) = \alpha_n^{1-d}\lambda_0^n A_0(d) z^d + \alpha_n r\frac{s}{t_n} + O(n\alpha_n \Lambda^{-n/(d-1)}\widetilde{\Lambda}^{-n}).$$

Setting $\alpha_n = (\lambda_0^n A_0(d))^{1/(d-1)}$ to normalize the coefficient of z^d, we have $|\alpha_n| \asymp \Lambda^{n/(d-1)}$ and thus:

$$\begin{aligned}
f_t^{1+n+p}(z) &= z^d + \alpha_n r\frac{s}{t_n} + O(n\widetilde{\Lambda}^{-n}) \\
&= z^d + \xi + O(\epsilon_n)
\end{aligned}$$

with $t = t_n(1 + \gamma_n\xi)$, γ_n and ϵ_n as in (3.2) and (3.3). Notice that if $|z|$ and $|\xi|$ are bounded by R in the expression above, then (3.11) is satisfied in our original coordinates. Reindexing n, we obtain the Theorem. ∎

4 Small Mandelbrot sets

We now show the Misiurewicz cascade leads to small Mandelbrot sets in parameter space. From this we deduce Theorems 1.1 and 1.3 of the Introduction.

Hybrid conjugacy. Let g_1, g_2 be polynomial-like maps of the same degree. A *hybrid conjugacy* is a quasiconformal map ϕ between neighborhoods of $K(g_1)$ and $K(g_2)$ such that $\phi \circ g_1 = g_2 \circ \phi$ and $\overline{\partial}\phi | K(g_1) = 0$. We say g_1 and g_2 are *hybrid equivalent* if such a conjugacy exists. By a basic result of Douady and Hubbard, every polynomial-like map g of degree d is hybrid equivalent to a polynomial of degree d, unique up to affine conjugacy if $K(g)$ is connected [DH, Theorem 1].

Theorem 4.1 *Let (f, c) be a degree d Misiurewicz bifurcation. Then the parameter space Δ contains quasiconformal copies $\mathcal{M}_d^n$ of the degree d Mandelbrot set M_d, converging to the origin, with $\partial\mathcal{M}_d^n$ contained in the bifurcation locus $B(f)$.*

More precisely, for all $n \gg 0$ there are homeomorphisms

$$\phi_n : M_d \to \mathcal{M}_d^n \subset \Delta$$

such that:

1. *f_t^n is hybrid equivalent to $z^d + \xi$ whenever $t = \phi_n(\xi)$;*
2. *$d(0, \mathcal{M}_d^n) \asymp |\lambda_0|^{-n/r}$;*
3. *$\mathrm{diam}(\mathcal{M}_d^n)/d(0, \mathcal{M}_d^n) \asymp |\lambda_0|^{-n/(d-1)}$;*
4. *ϕ_n extends to a quasiconformal map of the plane with dilatation bounded by $1 + O(\epsilon_n)$; and*
5. *$\psi_n^{-1} \circ \phi_n(\xi) = \xi + O(\epsilon_n)$, where $\psi_n(\xi) = t_n(1 + \gamma_n \xi)$.*

The notation is from (3.1) – (3.3).

We begin by recapitulating some ideas from [DH]. Let $\Delta(R) = \{z : |z| < R\}$, and let

$$g_\xi(z) = z^d + \xi + h(\xi, z)$$

be a holomorphic family of mappings defined for $(\xi, z) \in \Delta(R) \times \Delta(R)$, where $R > 10$ and $g'_\xi(0) = 0$. Let $\mathcal{M} \subset \Delta(R)$ be the set of ξ such that the forward orbit $g_\xi^n(0)$ remains in $\Delta(R)$ for all $n > 0$.

Lemma 4.2 *There is a $\delta > 0$ such that if $\sup |h(\xi, z)| = \epsilon < \delta$ then there is a homeomorphism*

$$\phi : M_d \to \mathcal{M}$$

such that for all $\xi \in M_d$, $g_{\phi(\xi)}$ is hybrid equivalent to $z^d + \xi$, $|\phi(\xi) - \xi| < O(\epsilon)$, and ϕ extends to a $1 + O(\epsilon)$-quasiconformal map of the plane.

Proof. Let $p_\xi(z) = z^d + \xi$. Since $R > 10$ we have $M_d \subset \Delta(R)$ and $K(p_\xi) \subset \Delta(R)$ for all $\xi \in M_d$; indeed these sets have capacity one, so their diameters are bounded by 4 [Ah]. In addition, for $\xi \in M_d$ the map $p_\xi : U \to \Delta(R)$ is polynomial-like, where $U = p_\xi^{-1}(\Delta(R))$. By continuity, when $\sup |h|$ is sufficiently small, $\mathcal{M}$ is compact and g_ξ is polynomial-like for all $\xi \in \mathcal{M}$.

By results of Douady and Hubbard, we can also choose δ small enough that $|h| < \delta$ implies there is a homeomorphism

$$\phi : M_d \to \mathcal{M}$$

such that $g_{\phi(\xi)}$ is hybrid equivalent to $z^d + \xi$ [DH, Prop. 21].

Now assume $|h| < \epsilon < \delta$. For $t \in \Delta$ let $\mathcal{M}_t$ denote the parameters where the critical point remains bounded for the family

$$g_{\xi,t} = z^d + \xi + t\frac{\delta}{\epsilon}h(\xi, z),$$

and define $\phi_t : M_d \to \mathcal{M}_t$ as above. Then ϕ_t is a family of injections, with $\phi_0(z) = z$, and $\phi_t(\xi)$ is a holomorphic function of t for every ξ. (For example this is clear at $\xi \in \partial M_d$ because Misiurewicz points are dense in ∂M_d; for the general case see [DH, Prop. 22].)

By a theorem of Słodkowski [Sl] (cf. [Dou], [BR]), $\phi_t(z)$ prolongs to a holomorphic motion of the entire plane, and its complex dilatation $\mu_t = \overline{\partial}\phi_t/\partial\phi_t$ gives a holomorphic map of the unit disk into the unit ball in $L^\infty(\widehat{\mathbb{C}})$. By the Schwarz lemma, $\|\mu_t\|_\infty \le |t|$; since $\phi = \phi_{\epsilon/\delta}$, we obtain a quasiconformal extension of ϕ with dilatation $1 + O(\epsilon)$. The bound on $|\phi(\xi) - \xi|$ similarly results by applying the Schwarz Lemma to the map $\Delta \to \widehat{\mathbb{C}} - \{0, 1, \infty\}$ given by $t \mapsto \phi_t(\xi)$, once three points have been normalized to remain fixed during the motion. ∎

Proof of Theorem 4.1. Fix $R > 10$. For all $n \gg 0$, Theorem 3.1 provides a family of rational maps of the form

$$g_\xi(z) = f_t^n(z) = z^d + \xi + O(\epsilon_n)$$

defined for $(\xi, z) \in \Delta(R) \times \Delta(R)$, where $t = \psi_n(\xi) = t_n(1 + \gamma_n\xi)$. The preceding Lemma gives homeomorphisms $\widetilde{\phi}_n : M_d \to \widetilde{\mathcal{M}_d^n} \subset \Delta(R)$ for all $n \gg 0$. Setting $\phi_n = \psi_n \circ \widetilde{\phi_n}$, the Theorem results from the Lemma and the bounds (3.1) – (3.3). ∎

Example. The quadratic family $(f, c) = (z^2 + t - 2, 0)$ is a Misiurewicz bifurcation of degree $d = 2$, with $\lambda_0 = 2$ and $r = 1$. Thus M_2 contains small copies $\mathcal{M}_2^n$ of itself near $c = -2$, with $d(\mathcal{M}_2^n, -2) \asymp 4^{-n}$ and $\operatorname{diam} \mathcal{M}_2^n \asymp 16^{-n}$.

Consequences. Assembling the preceding results, we may now prove the Theorems stated in the Introduction. Here is a more precise form of Theorem 1.1:

Theorem 4.3 *Let f be a holomorphic family of rational maps over the unit disk with bifurcations. Then there is a nonempty list of degrees*

$$D \subset \{2, 3, \dots, 2^{2\deg(f_t)-2}\}$$

such that for any $\epsilon > 0$ and $d \in D$, $B(f)$ contains the image of ∂M_d under a $(1+\epsilon)$-quasiconformal map.

If the critical points of f are marked $\{c_t^1, \dots, c_t^m\}$ such that

$$(i \leq N) \iff c^i \textit{ is active and } c^i_t \textit{ is unramified for some } t,$$

then we may take

$$D = \{\inf_t \sup_k \deg(f^k_t, c^i_t) \; : \; i \leq N\}.$$

Proof. Let $B_0 = B(f)$. After a base change we can assume that f is a local bifurcation with critical points marked as above. By Proposition 2.4, there is at least one active, unramified critical point, so $N \geq 1$. For any $i \leq N$, we can make a base change so c^i_t is active and unramified; then by Proposition 2.5, a further base change makes (f^n, c^i) into a degree d Misiurewicz bifurcation.

Let $d_i = \inf_t \sup_k \deg(f^k_t, c^i_t)$. We claim $d = d_i \leq 2^{2\deg(f_t)-2}$. Indeed, $\deg(f^n_t, c^i_t)$ assumes its minimum outside a discrete set, and it is equal to d near $t = 0$, so $d_i \geq d$. On the other hand, c^i_0 lands on a repelling periodic cycle, so $\deg(f^k_0, c^i_0) = d$ for all $k > n$, and therefore $d_i \leq d$. Finally d is largest if c^i hits all the other critical points of f before reaching the repelling cycle; in this case $d = (p_1{+}1)(p_2{+}1)\cdots(p_m{+}1)$ for some partition $p_1{+}p_2{+}\cdots{+}p_m = 2d{-}2$. The product is maximized by the partition $1 + 1 + \cdots + 1$, so $d \leq 2^{2\deg(f_t)-2}$.

By Theorem 4.1, the bifurcation locus $B(f)$ contains almost conformal copies $\partial\mathcal{M}^n_d$ of ∂M_d accumulating at $t = 0$, with $\operatorname{diam}(\mathcal{M}^n_d) \ll d(0, \mathcal{M}^n_d)$. Letting $\phi : \Delta \to \Delta$ denote the composition of all the base-changes occurring so far, we have $B(f) = \phi^{-1}(B_0)$. Then ϕ is univalent and nearly linear on $\mathcal{M}^n_d$ for $n \gg 0$, so $\phi(\partial\mathcal{M}^n_d) \subset B_0$ is a $(1 + \epsilon)$-quasiconformal copy of ∂M_d. ■

Let Rat_d be the space of all rational maps of degree d; it is a Zariski-open subset of $\mathbb{P}^{2d+1}$. We now make precise the statement that a generic family contains a copy of the standard Mandelbrot set.

Corollary 4.4 *There is a countably union of proper subvarieties $R \subset \mathrm{Rat}_d$ such that for any local bifurcation, either $f_t \in R$ for all t, or $B(f)$ contains a copy of ∂M_2.*

Proof. On a finite branched cover X of Rat_d, the critical points of $f \in \mathrm{Rat}_d$ can be marked $\{c^1(f), \ldots, c^{2d-2}(f)\}$. Clearly $\deg(f^n, c_i(f)) = 2$ outside a proper subvariety $V_{n,i}$ of X. Let R be the union of the images of these varieties in Rat_d, and apply the preceding argument to see $D = \{2\}$ if some $f_t \notin R$. ■

Proof of Theorem 1.3. The proof follows the same lines as that of Theorem 4.3; to get $\mathrm{mod}(V{-}U)$ large one takes R large in Theorem 3.1. Thus Theorem 1.3 also holds for all $d \in D$. ■

5 Hausdorff dimension

In this section we prove Corollary 1.8: for any holomorphic family f of rational maps over a complex manifold X, we have $\mathrm{H.dim}(B(f)) = \mathrm{H.dim}(X)$ if $B(f) \neq \emptyset$.

Recall that the *Hausdorff dimension* of a metric space X is the infimum of the set of $\delta \geq 0$ such that there exists coverings $X = \bigcup X_i$ with $\sum (\operatorname{diam} X_i)^\delta$ arbitrarily small.

Lemma 5.1 *Let Y be a metric space, X a subset of $Y \times [0,1]$. Then*

$$\mathrm{H.dim}(X) \geq 1 + \inf \mathrm{H.dim}(X_t)$$

where $X_t = \{y \,:\, (y,t) \in X\}$.

Here $Y \times [0,1]$ is given the product metric.

Proof. Fix δ with $\delta + 1 > \mathrm{H.dim}(X)$. For any n there is a covering $X \subset \bigcup B(y_i, r_i) \times I_i$ with $|I_i| = r_i$ and $\sum r_i^{\delta+1} < 4^{-n}$. Note that

$$X_t \subset \bigcup_{t \in I_i} B(y_i, r_i)$$

and

$$\int_0^1 \sum_{t \in I_i} r_i^\delta \, dt \;=\; \sum r_i^{\delta+1} < 4^{-n}.$$

Let E_n be the set of t where the integrand exceeds 2^{-n}; then $\sum m(E_n) < \sum 2^{-n} < \infty$. Thus almost every t belongs to at most finitely many E_n, so almost every X_t admits infinitely many coverings with $\sum r_i^\delta < 2^{-n} \to 0$. Therefore $\delta \geq \inf \mathrm{H.dim}(X_t)$, and the Theorem follows. ■

The Lemma above is related to the product formula

$$\mathrm{H.dim}(X \times Y) \geq \mathrm{H.dim}(X) + \mathrm{H.dim}(Y);$$

see [Fal, Ch. 5] and references therein.

Proof of Corollary 1.8. Suppose $B(f) \neq \emptyset$. Then there is a $t_0 \in B(f)$ and a locally parameterized periodic point $a(t)$ of period n such that $a(t)$ changes from attracting to repelling near t_0 [MSS], [Mc2, §4.1]. More formally this means the multiplier $\lambda(t) = (f^n)'(a(t))$ is not locally constant and $|\lambda(t_0)| = 1$.

Choosing local coordinates we can reduce to the case $X = \Delta^n$ and $t_0 = 0$. Let $\Delta_s = \Delta \times \{s\}$ for $s \in \Delta^{n-1}$. For coordinates in general position, $\lambda(t)$ is nonconstant on Δ_0. Shrinking the Δ^{n-1} factor, we can also assume $a(t)$ changes from attracting to repelling in the family $f|\Delta_s$ for all s. Then

$$B(f)_s = B(f) \cap \Delta_s \supset B(f|\Delta_s) \neq \emptyset$$

and $\mathrm{H.dim}\, B(f|\Delta_s) = 2$ by Corollary 1.6. Applying the Lemma above to $B(f) \subset \Delta \times \Delta^{n-1}$ we find

$$\mathrm{H.dim}(B(f)) \geq (2n-2) + \inf_s \mathrm{H.dim}\, B(f)_s = 2n = \mathrm{H.dim}(X).$$

∎

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